The authors concede that, in principle, aimpoint wander may exist due to either intensity modulation or angle wander. In that connection, the criticism of the discussion of the observations as illustrated in Fig. 6 is justified, in that the spikes in the error signal record marked e, f, and g may truly be aimpoint excursions due to interference of echoes from the target, and that the echoes need not necessarily originate from the terrain just because the direction from which they appear to come falls off the target.

It is also true, however, that the error signals in the data presented in the paper are strictly products of angular error and signal intensity. It follows that the spikes in the error signal record may be due not to an aimpoint wander, in the sense of Mr. Howard's suggestion, but only to a sudden increase in echo intensity, as is our conclusion. The primary datum in support of this conclusion is that the spikes in the error signal record are always quickly damped. This is reasonably concluded to be due to the use of the AGC, if the cause of the spike is a sudden increase in signal intensity. There is no reason to expect such sudden damping if the spike is a true aimpoint wander.

We thank Mr. Howard for his helpful and thoughtful comments.

Comment on "Optimizing the Gains of the Baro-Inertial Vertical Channel"

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Introduction

IN the paper "Optimizing the Gains of the Baro-Inertial Vertical Channel" by Widnall and Sinha,[1] the selection of the three vertical channel gains was formulated as a stochastic optimal control problem, where the objective was to minimize the mean-square error of the indicated vertical velocity. A fifth-order error model was the basis for the analysis. An analytical expression was found for the mean-square vertical velocity error as a function of the assumed statistics of the sources of error and of the undetermined gains. A computer program was used to find the set of gain values that minimizes the mean-squared error. Sensitivity of the results to the statistical assumptions was explored. Analysis of the numerical results led to the discovery of approximate analytical formulas giving the optimal gains and pole locations as a function of the assumed statistics of the sources of error.

This note points out that the principal results of the paper could have been obtained without the use of a numerical minimum-finding computer program. This eliminates the usual concern associated with using such a program—namely, that the program may have only found a local minimum. It is shown that there is a third-order optimal estimator whose estimation error is governed by the same differential equation as that governing a third-order subsystem of the vertical channel. The results from optimal estimation theory therefore can be used for the vertical channel optimization problem. The poles of the optimal estimator (and the optimized vertical channel) are governed by a symmetric root characteristic equation. Having determined the optimal poles, the optimal gains may be calculated from the pole locations. This note also points out that the approximate analytical formulas for the optimal pole locations may be derived directly from the symmetric root characteristic equation.

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Related Optimal Estimation Problem

The fifth-order error model used in the paper and in this note is

$$\delta\dot{h} = \delta v_z + u_1 \qquad \delta\dot{v}_z = c\delta h - \delta\hat{a} + \delta a + u_2 + w_{a1}$$
$$\delta\dot{\hat{a}} = -u_3 \qquad \delta\dot{a} = w_{a2} \qquad \delta\dot{b} = w_{b2} \tag{1}$$

where δh is the error in indicated altitude, δv_z is the error in indicated vertical velocity, $\delta\hat{a}$ is the computed vertical acceleration error, δa is the slowly varying acceleration error, and δb is the slowly varying error in altitude indicated by the barometric altimeter. w_{a1}, w_{a2}, w_{b2} are white noises of spectral density Q_{a1}, Q_{a2}, Q_{b2} which provide the short correlation time acceleration error, the acceleration error random walk, and the altimeter error random walk. u_1, u_2, u_3 are control variables. The constant c is the gravity gradient constant whose value near the surface of the earth is $c = 2g/R = 3.07 \times 10^{-6}\ \mathrm{s}^{-2}$. The measurement, from which are derived the corrections to the vertical channel variables, is the difference between the indicated altitude and the altitude indicated by the barometric altimeter. In terms of the errors in these variables, the measurement is

$$y = \delta h - \delta b - w_{b1} \tag{2}$$

where w_{b1} is the white noise of the spectral density Q_{b1} modeling short correlation time altimeter error. In the baro-inertial vertical channel, the control variables are simply

$$u_1 = -k_1 y \qquad u_2 = -k_2 y \qquad u_3 = -k_3 y \tag{3}$$

It can be shown the fifth-order state vector is not completely observable through the measurement y. A bias acceleration error δa offset by a bias computed acceleration error $\delta\hat{a}$ cannot be observed through the measurement y, and a bias altimeter error δb offset by a bias indicated altitude δh cannot be observed through the measurement y. A third-order subsystem that is completely observable has these state variables:

$$x_1 = \delta h - \delta b \qquad x_2 = \delta v_z \qquad x_3 = -\delta\hat{a} + \delta a + c\delta b \tag{4}$$

This subsystem and the measurement are governed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ c & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & c \end{bmatrix} \begin{bmatrix} w_{a1} \\ w_{a2} \\ w_{b2} \end{bmatrix} \tag{5}$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - w_{b1} \tag{6}$$

Equations (3), (5), and (6) are seen to be in the form

$$u = -Ky \tag{7}$$

$$\dot{x} = Fx + u + Gw \tag{8}$$

$$y = Mx + v \tag{9}$$

Combining Eqs. (7-9), the differential equation governing the observable subsystem of the vertical channel is

$$\dot{x} = (F - KM)x + Gw - Kv \tag{10}$$

Of interest is the optimal estimator of the state of the subsystem governed by Eqs. (8) and (9). The well-known Kalman-Bucy filter which provides the optimal estimate $\hat{x}$ is governed by

$$\dot{\hat{x}} = F\hat{x} + u + K(y - M\hat{x}) \tag{11}$$

If we define the estimation error to be

$$e = x - \hat{x} \tag{12}$$

by subtracting Eq. (11) from Eq. (8) and using Eq. (9), it can be shown that the estimation error is governed by

$$\dot{e} = (F - KM)e + Gw - Kv \tag{13}$$

The estimation error is governed by the same differential equation as that governing the observable subsystem of the vertical channel, Eq. (10). Hence, if we find the steady-state estimator gain vector K that minimizes the mean-square estimation error, then we have also found the control gain vector K that minimizes the mean-square values of the th--- states of the observable subsystem in the vertical channel.

Symmetric Root Characteristic Equation

Without first calculating the optimal gains of the estimator, one can obtain the natural frequencies (poles) of the steady-state optimal estimator from its symmetric root characteristic equation. The key theorem is that if the system $\{F,G\}$ is stabilizeable (it is) and if the system $\{F,M\}$ is detectable (it is), then the poles of the steady-state optimal estimator are the left-half plane roots of the symmetric root characteristic equation[2]

$$\phi(s)\phi(-s)[I + R^{-1}H(s)QH^T(-s)] = 0 \tag{14}$$

where $H(s)$ is the noise-vector-to-measurement transfer matrix

$$H(s) = M[sI - F]^{-1}G \tag{15}$$

and $\phi(s)$ is the characteristic polynomial of the system

$$\phi(s) = \det[sI - F] \tag{16}$$

R is the spectral density of the measurement noise

$$R = Q_{b1} \tag{17}$$

and Q is the spectral density matrix of the state driving noise vector

$$Q = \mathrm{diag}[Q_{a1}, Q_{a2}, Q_{b2}] \tag{18}$$

Carrying out the indicated operations, one finds that the symmetric root characteristic equation is

$$s^2(s^2 - c)^2 Q_{b1} - (s^2 - c)^2 Q_{b2} + s^2 Q_{a1} - Q_{a2} = 0 \tag{19}$$

To find the roots of this sixth-order equation, note that if one lets $r = s^2$, one has a cubic equation in r. There are known formulas for the roots of such a cubic equation, so one does not need to use a numerical root-finder program. If r_1, r_2, r_3 are the three complex roots of the cubic equation, then the six complex roots of the sixth-order equation are the plus and minus complex square roots of r_1, r_2, r_3. Denote the three left-half plane roots p_1, p_2, p_3. These are the poles of the steady-state optimal estimator.

Given the poles of the optimal estimator, the characteristic equation of the optimal estimator must be

$$(s - p_1)(s - p_2)(s - p_3) = 0 \tag{20}$$

or

$$s^3 - (p_1 + p_2 + p_3)s^2 + (p_1p_2 + p_1p_3 + p_2p_3)s - p_1p_2p_3 = 0 \tag{21}$$

The fundamental matrix of the estimation error differential equation (13) is $F - KM$, so the characteristic equation in terms of the gains is

$$\det[sI - F + KM] = 0 \tag{22}$$

or

$$s^3 + k_1s^2 + (k_2 - c)s + k_3 = 0 \tag{23}$$

Since Eqs. (21) and (23) are the same characteristic equation, the gains must be

$$\begin{aligned} k_1 &= -(p_1 + p_2 + p_3) \\ k_2 &= c + p_1p_2 + p_1p_3 + p_2p_3 \\ k_3 &= -p_1p_2p_3 \end{aligned} \tag{24}$$

Equation (24) may be used to calculate the optimal gains as explicit functions of the poles of the steady-state optimal estimator.

Approximate analytical formulas for the optimal pole locations, as functions of the assumed statistics of the sources of error, may be derived directly from the symmetric root characteristic equation (19). If the acceleration error noise densities Q_{a1} and Q_{a2} are comparatively small, Eq. (19) is approximately

$$(s^2 - c)^2(s^2Q_{b1} - Q_{b2}) = 0 \tag{25}$$

The three left-half plane roots p_1, p_2, p_3 are

$$p_1 = -\sqrt{Q_{b2}/Q_{b1}} \tag{26}$$

$$p_2, p_3 = -\sqrt{c} \tag{27}$$

Alternatively, if the noise density Q_{b1} of the short correlation time altimeter error is comparatively small, two of the six roots of Eq. (19) go to comparatively large values of s governed by

$$s^6Q_{b1} - s^4Q_{b2} = 0 \tag{28}$$

The left-half plane root p_1 is again given by Eq. (26). Four of the six roots remain at comparatively small values of s governed by

$$(s^2 - c)^2Q_{b2} - s^2Q_{a1} + Q_{a2} = 0 \tag{29}$$

This is a quadratic equation in the variable $r = s^2$, so it is easily solved. After some arithmetic manipulation, one obtains for the left-half plane roots p_2, p_3

$$p_2,p_3 \approx -\frac{1}{2}\sqrt{2c+\frac{Q_{a1}}{Q_{b2}}+2\sqrt{c^2+\frac{Q_{a2}}{Q_{b2}}}}$$
$$\pm\frac{1}{2}\sqrt{2c+\frac{Q_{a1}}{Q_{b2}}-2\sqrt{c^2+\frac{Q_{a2}}{Q_{b2}}}} \quad (30)$$

which is as given earlier in the paper.[1]

Comparison with Optimal Controller

It is interesting to note that the vertical channel is optimal in a larger sense. In the vertical channel, the control variables were constrained to be the simple functions of the measurement variable given in Eq. (3). Can a more complicated compensation connecting the measurement and control variables provide smaller mean-square errors? It is well known that optimal control is provided by feedback compensation that is the cascade combination of the Kalman-Bucy filter and the state-feedback optimal regulator.[3] The filter was given in Eq. (11). The regulator is

$$u = -C\hat{x} \quad (31)$$

In this problem there is no penalty associated with using large values for the control variables, so the optimal regulator gains become arbitrarily large. As a result, the estimate $\hat{x}$ is held near zero. Now it is known that under optimal control[3]

$$X = \hat{X} + P \quad (32)$$

where $X = E[xx^T]$, $\hat{X} = E[\hat{x}\hat{x}^T]$, $P = [ee^T]$. With $\hat{x}$ held near zero, the covariance of the state under optimal control is the same as the covariance of the optimal estimation error:

$$X = P \quad (33)$$

We previously established that the state of the simple closed-loop vertical channel is governed by the same differential equation, as is the estimation error of the Kalman-Bucy filter. So, for the simple vertical channel, Eq. (33) is also true. Therefore, the simple vertical channel achieves the same optimal performance as the more complicated unrestricted optimal controller.

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Comment on "Observer Stabilization of Singularly Perturbed Systems"

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IN Ref. 1 we showed that, for linear time-invariant singularly perturbed systems, a controller consisting of a linear feedback law and a Luenberger observer, both based on the reduced-order model obtained when the small parameter ϵ is set to zero, would stabilize the full-order system (and, in fact, the state estimate would converge to the appropriate part of the full system state) when ϵ is positive and sufficiently small. Although several methods of proof exist, the one used in the above note relied upon the well-known Klimushchev-Krasovskii lemma. An application of the result for autopilot-actuator design was presented.

Recently, it has come to our attention that the above result was, in fact, proved before by B. Porter[2]; however, the autopilot-actuator application does not seem to have appeared before. Although our work was quite independent of that in Ref. 2, Porter did publish the result first in 1974, and we wish to acknowledge the priority of his work.

We hope this note will help to set the historical record straight.

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